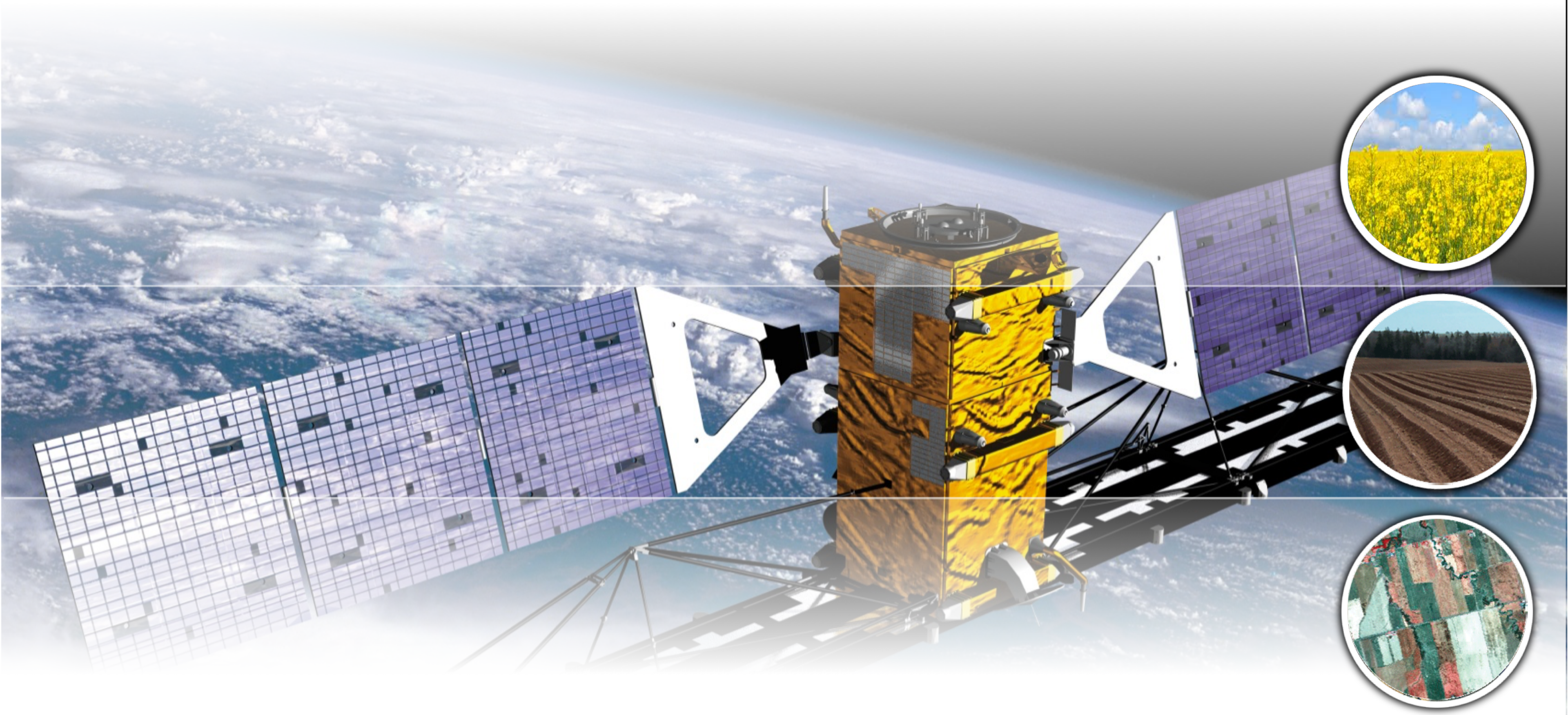


Lecture 3:
Matrix Algebra for Geographers

Structure

- Matrix calculation - Basics
- Complex numbers

Matrix - Basics

- Definition:
  - Matrix: rectangular array of numbers arranged in m rows and n columns
  - Vector: matrix with 1 row or 1 column

- Notation:
  - Matrix $\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$
  - Column vector $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$
  - Row vector $\mathbf{x'} = \left[x_1 \ldots x_n \right]$

Matrix - Basics

- Transposed matrix $\mathbf{A}^T$
  - Matrix is mirrored at the main diagonal.

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \rightarrow \mathbf{A}^T = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix}$$

Matrix Operations (I)

- Addition and subtraction: two matrices **A** and **B** are summed / subtracted by adding their elements at the same positions.

$$\mathbf{C} = \mathbf{A} \pm \mathbf{B}: \quad c_{ik} = a_{ik} \pm b_{ik} \quad \text{for } i = 1, \ldots, m, \; k = 1, \ldots, n$$

Matrix Operations (II)

- Multiplication of a matrix with a scalar
  - A matrix **A** is multiplied with a scalar by multiplying every element of **A** with c.

$$\mathbf{B} = c \cdot \mathbf{A}: \quad b_{ik} = c \cdot a_{ik} \quad \text{for } i = 1, \ldots, m, \; k = 1, \ldots, n$$

Matrix Operations (III)

- Multiplication of matrices
  - The product **C** = **AB** of (m x n) matrix **A** with (n x p) matrix **B** is defined as

$$c_{ik} = \sum_{r=1}^{n} a_{ir} \cdot b_{rk} \quad \text{for} \quad i = 1, \ldots, m, \quad k = 1, \ldots, p$$

  - Pre-condition: numbers of columns of **A** = number of rows of **B**

  - **Note:** matrix multiplication is not commutative, i.e. **AB** ≠ **BA**

Eigenvalues and Eigenvectors (I)

- Think of eigenvalues and eigenvectors as providing Summary of a large matrix
- They are used to reduce dimension space.
- If A is an $n \times n$ matrix, do there exist nonzero vectors $\mathbf{x}$ in R^n such that $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$?

A: an $n \times n$ matrix

λ: a scalar (could be **zero**)

$\mathbf{x}$: a **nonzero** vector in R^n

Eigenvalue

$$A\mathbf{x} = \lambda\mathbf{x}$$

Eigenvector

Eigenvalues and Eigenvectors (II)

- Example:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Eigenvalue

$$A\mathbf{x}_1 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\mathbf{x}_1$$

Eigenvector

Eigenvalue

$$A\mathbf{x}_2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = (-1)\mathbf{x}_2$$

Eigenvector

Complex numbers (I)

Motivation: polynoms of n-th degree should have n zeros.

$$z^2 - 1 = 0 \quad \Rightarrow \quad (z-1)(z+1) = 0 \quad \Rightarrow \quad z_1 = 1; \quad z_2 = -1$$

$$z^2 = 0 \quad \Rightarrow \quad z_1 = 0; \quad z_2 = 0$$

$$z^2 + 1 = 0 \quad \Rightarrow \quad z_1 = \sqrt{-1}; \quad z_2 = -\sqrt{-1}$$

Complex numbers (II)

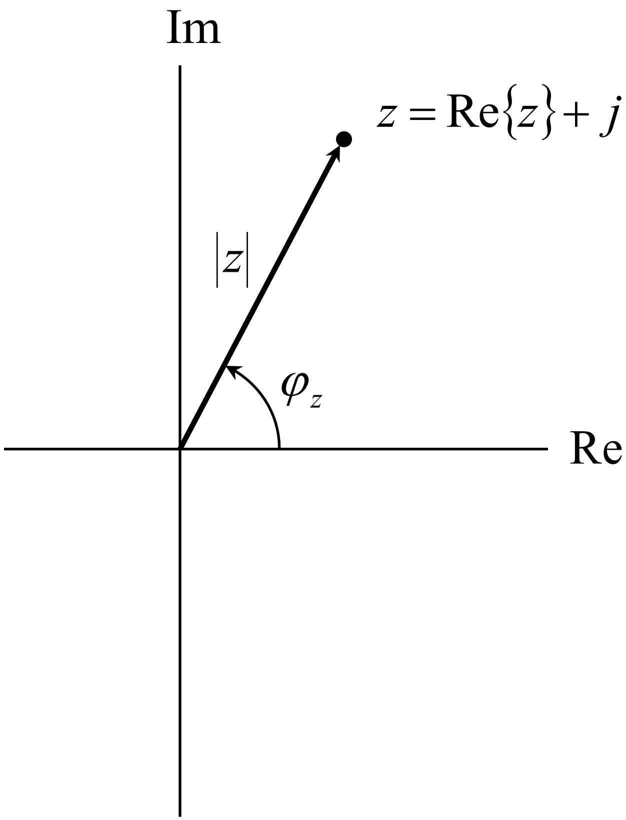
Def.: $j = \sqrt{-1}$ Imaginary unit (mostly named „ *i* “)

$z = x + j\,y \qquad x, y \in \Re$ Complex number

$x = \mathrm{Re}\{z\}$ Real part

$y = \mathrm{Im}\{z\}$ Imaginary part

Complex numbers (III)



$$z = \mathrm{Re}\{z\} + j\,\mathrm{Im}\{z\} = |z|\exp(j\varphi_z)$$

$$|z| = \sqrt{\mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2}$$

$$\varphi_z = \arctan\left(\frac{\mathrm{Im}\{z\}}{\mathrm{Re}\{z\}}\right)$$

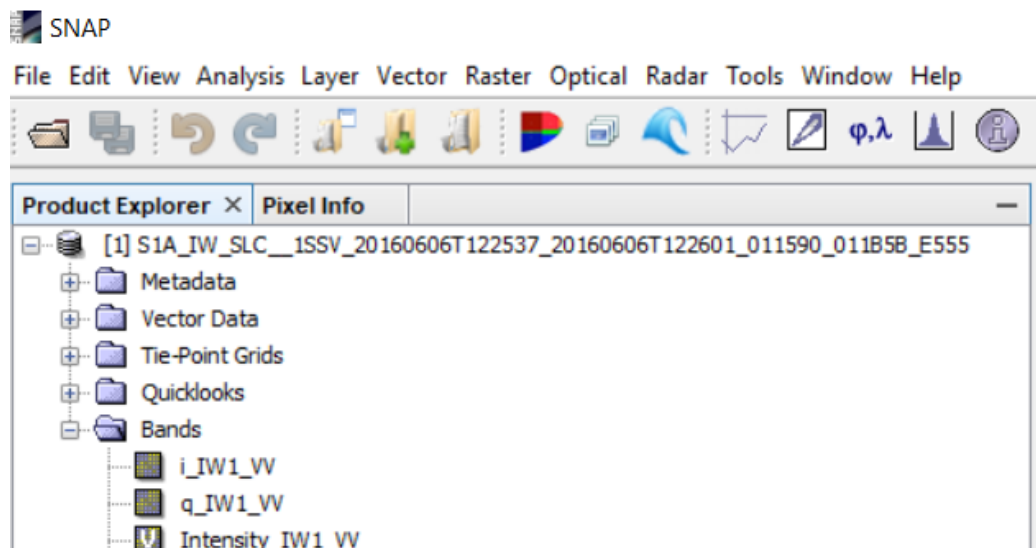


Image source: https://eo-college.org/

Complex numbers (IV)

- Complex conjugate

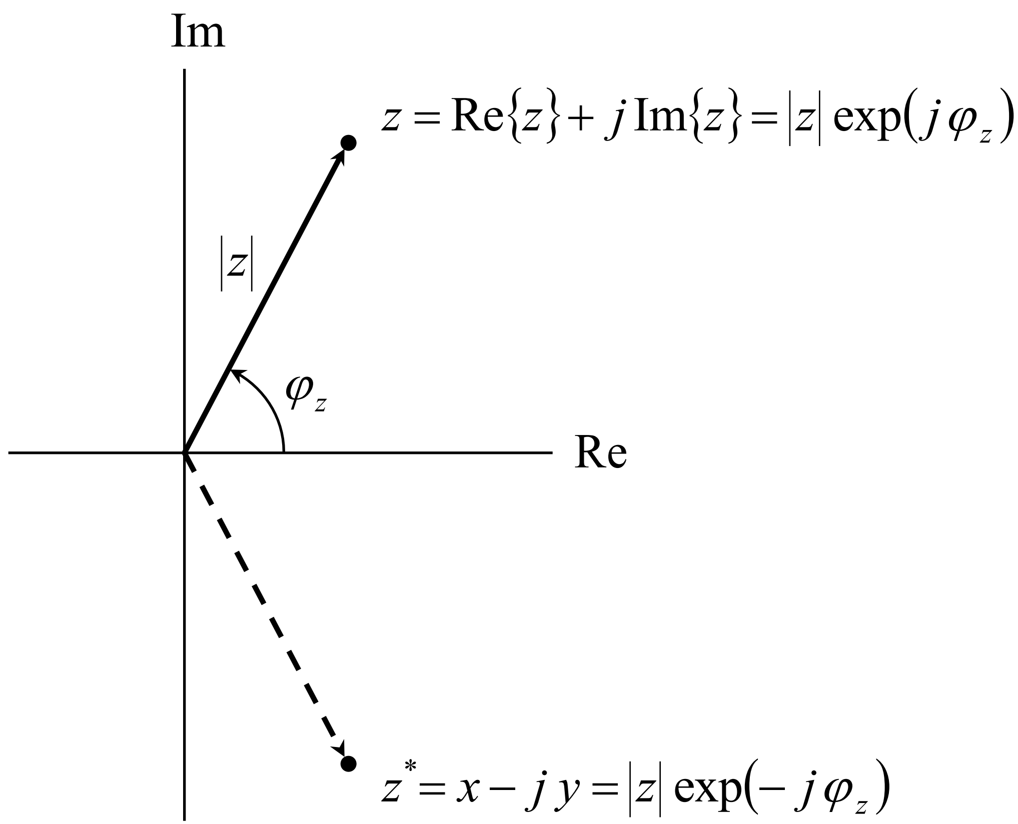


Image source: https://eo-college.org/

Complex numbers (V)

Summation: $z_1 + z_2 = \mathrm{Re}\{z_1\} + \mathrm{Re}\{z_2\} + j\left(\mathrm{Im}\{z_1\} + \mathrm{Im}\{z_2\}\right)$

Corresponds to vector summation:

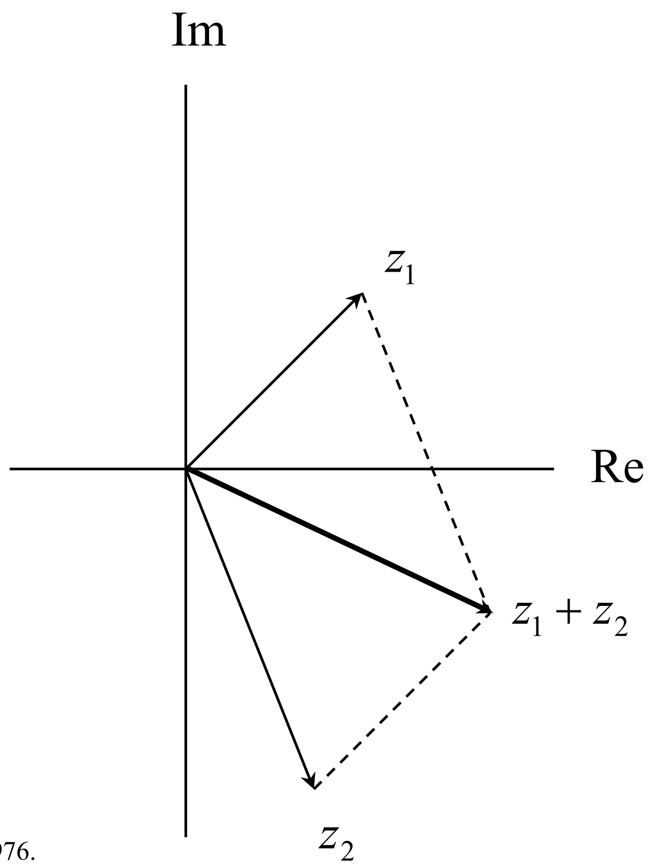
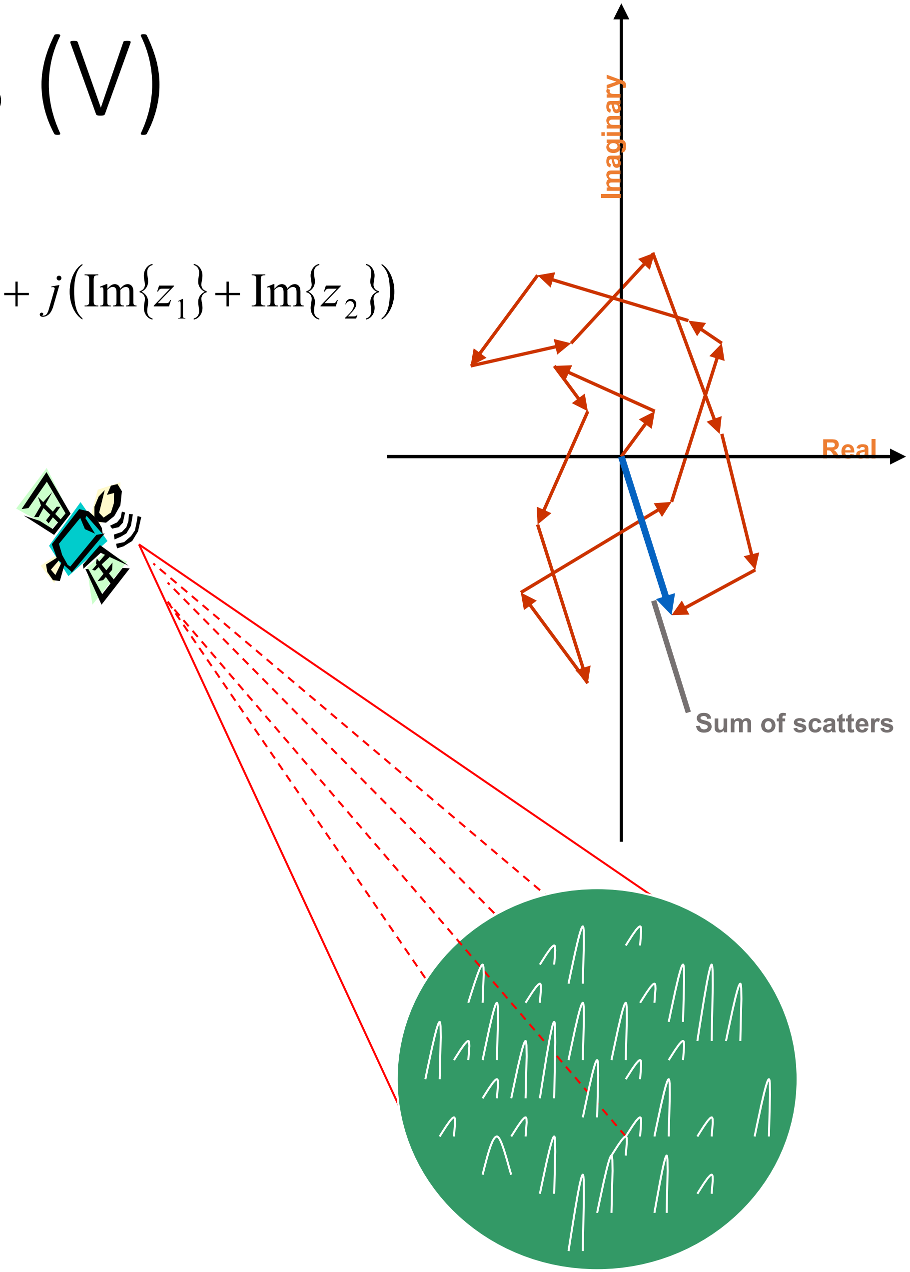


Image source: Goodman, 1976.



Complex numbers (VI)

Multiplication: $z_1 \cdot z_2 = |z_1|\,|z_2| \exp\left(j\left(\varphi_{z_1} + \varphi_{z_2}\right)\right)$

$$z \cdot z^* = |z|^2$$

Life is a complex number!

It has a real part and an imaginary part!

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